Pricing American[1] Options

Theory and Practice

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1 actually Bermudan!

2 This talk represents the views of the author alone, and not the views BofA Securities or any of her previous employers.

1. Introduction
2. American Options
   - Definitions
   - Intuition
3. Pricing
   - Least Squares Monte Carlo
   - Alternative Methods

Introduction

Questions and Takeaways

- What is a Quant?
- What is an American option? and why do we care about them?
- What is the difficulty of pricing American options? Why numerical methods are needed?
- Becoming a Quant is a cool career option that you may want to consider
- As a Quant you will have fun and make use of the tools aquired during your studies. Particularly from Mathematics, Statistics, and Programming
- Today we will discuss and implement one method to price American options

What is a Quant?

From Wikipedia

- Quantitative analysis is the use of mathematical and statistical methods[a] in finance and investment management. Those working in the field are quantitative analysts (quants).
- Although the original quantitative analysts were "sell side quants" from market maker firms, concerned with derivatives pricing and risk management, the meaning of the term has expanded over time to include those individuals involved in almost any application of mathematical finance, including the buy side.

[a] I would add computational/programming methods!

What is a Quant?

From ChatGPT

- A quant is a colloquial term for a quantitative analyst, someone who applies mathematical and statistical methods[a] to financial and risk management problems. Quants work primarily in the finance industry, using their expertise to develop models and algorithms for trading, investment strategies, risk assessment, and other financial applications.
- They often have backgrounds in fields such as mathematics, statistics, physics, computer science, or engineering.

[a] again I would also stress the computational part

Quants in IB - How it Started

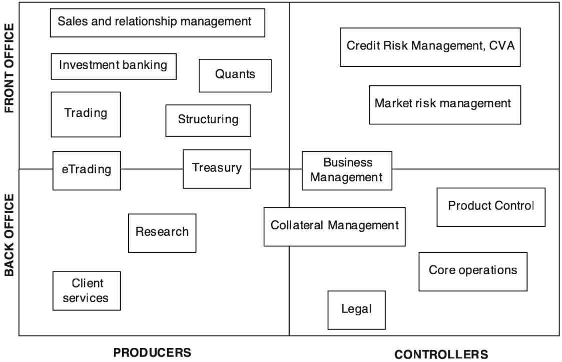

Figure 1.1 Divisions of an investment bank

Figure: Taken from "The Front Office Manual: The Definitive Guide to Trading, Structuring and Sales (Global Financial Markets) (2013)" by A. Sutherland, and J. Court.

Quant in IB - How it's Going

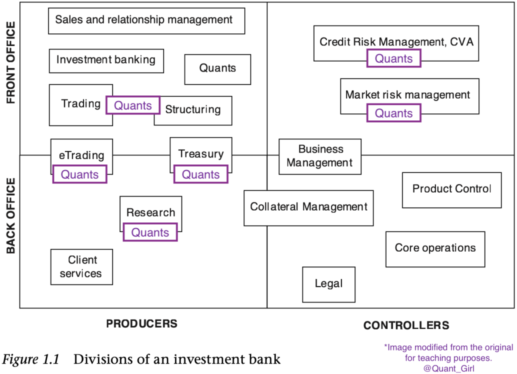

Figure 1.1 Divisions of an investment bank

Figure: Modified from the original version which appeared in "The Front Office Manual: The Definitive Guide to Trading, Structuring and Sales (Global Financial Markets) (2013)" by A. Sutherland, and J. Court.

American Options

Options

- An option is a contract which gives the buyer/holder the right, but not the obligation, to buy or sell a particular financial product known as the option's underlying instrument/asset.
- The underlying asset could be stocks, bonds, commodities, currencies, or other derivatives.
- In Equities, the underlying is typically a stock, an ETF, or other similar product.
- The contract establishes a specific price, called the strike price, at which the contract may be exercised, or acted upon.
- It also sets an expiration date. When an option expires, it no longer has value and no longer exists.

American Options

Definition

A contract between two parties giving the buyer the right to sell (purchase) one unit of a security for the amount K at any time on or before a fixed maturity T.

In contrast, European options can only be exercised at maturity T.

Options Market Share

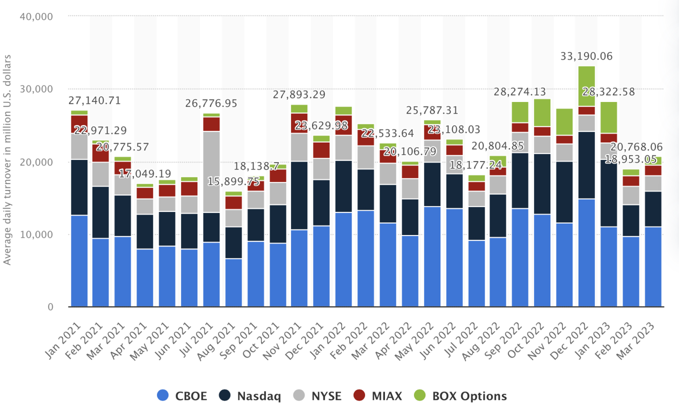

Figure: Average daily turnover value of exchange traded options in the United States from January 2021 to March 2023, by exchange.

Options Market Share

- In total, the average daily turnover of exchange traded options in January 2023 was approximately 20.77 billion U.S. dollars.
- Between January 2021 and March 2023, the Chicago Board Options Exchange (CBOE) was consistently the largest market for options traded in the United States, accounting for almost 62% of the average daily exchange traded options during March 2023.

General Properties

- An American option can only be exercised once
- The buyer of the option has the choice of when to do so. Moreover, the exercise decision can only be based on the information up to the present moment
- American options are more valuable than their European counterparts. Can you guess why? —Note that this is only true if the risk free rate is positive.

Intuition

Let's take a look at some simulations to gain more intuition about the problem!

Pricing

How to Price American Options

- Unlike European options where closed form pricing formula are available, pricing American options is complicated due to their exercise contingency.
- Numerical methods are always needed
- Today we will focus on the so-called Least-Squares Monte Carlo or Longstaff-Schwartz method
- I do recomend to take a look at the original paper

Mathematical Formulation

- $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ a filtered probability space
- Let X denote the price of a risky asset whose dynamics follow a stochastic process $X = \{X(t) : 0 \leq t \leq T\}$
- Let B denote the process representing the money market account, i.e. $B = \{B(t) : 0 \leq t \leq T\}$, where

$$B(t) = \exp\left(\int_0^t r_u du\right),$$

  and r is the instantaneous short rate process.
- Consider an American put option for the underlying X with strike K and maturity T

Mathematical Formulation
Holder's Point of View

- At maturity T the value of the option equals the payoff

$$Y(T) = (K - X(T))^+ = \max\{K - S(T), 0\}.$$

- At any time $t \in [0, T)$ the option buyer has two choices
  1. Exercise: this gives immediate payoff

  $$Y(t) = (K - X(t))^+,$$

  2. Hold/Continue: he will keep the right to exercise in the remaining time (t, T). This leaves him with a derivative with hold/continue value, i.e. the price of the option if not exercised until time t.
- Note that if $K \le X(t)$, the buyer has no real incentive to exercise and most lilekly will chose to hold.
- If $K > X(t)$, then the right choice is not clear. The exercise time is a stopping time.

Mathematical Formulation
Seller's Point of View

- If the option seller knew in advance which stopping time τ_0 the investor will use, then he would set the price as

$$\mathbb{E}_Q\left[\frac{Y(\tau_0)}{B(\tau_0)}\right].$$

- However, the optimal stopping time is not know. Thus, the option seller has to prepare for the worst possible case and charge the maximum value, i.e.:

$$\sup_{\tau\in\mathcal{T}}\mathbb{E}_Q\left[\frac{Y(\tau)}{B(\tau)}\right],$$

where $\mathcal{T}$ is the class of admisible stopping times taking values in $[0, T]$.

Solution by Using Snell Envelope

Proposition 1

Define $Z = \{Z(t) : 0 \le t \le T\}$, given by

$$Z(t) = \sup_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}_Q \left[\frac{Y(\tau)}{B(\tau)} \middle| \mathcal{F}_t \right] B(t).$$

Then $Z(t)/B(t)$ is the smallest $\mathbb{Q}$-supermartingale satisfying $Z(t) \ge Y(t)$. Moreover, the supremum in section 1 is achieved by an optimal stopping time of the form

$$\tau(t) = \inf \{ s \ge t : Z(s) = Y(s) \}.$$

That is, $\tau(t)$ maximises the right hand side of section 1:

$$\mathbb{E}_Q \left[\frac{Y(\tau)}{B(\tau)} \middle| \mathcal{F}_t \right] = \sup_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}_Q \left[\frac{Y(\tau)}{B(\tau)} \middle| \mathcal{F}_t \right].$$

Ideas in the Proof

- The key idea is to work backwards in time constructing $Z(t)$ by using dynamic programming
- More precisely, to introduce a process $U = \{U(t) : 0 \leq t \leq T\}$ given by

$$U(t) := \begin{cases} Y(t) & t = T, \\ \max\left\{ Y(t), \underbrace{\mathbb{E}_{\mathbb{Q}}\left[\frac{U(t+1)}{B(t+1)} \middle| \mathcal{F}_t \right] B(t)}_{\text{Expected Payoff from Hold/Continue}} \right\} & t \leq T-1, \end{cases}$$

- This process is called snell envelope and is the smallest supermartingale dominating the process Y
- It then follows that $Z = U$.

In Practice we are trying to find Rules to Stop/Exercise

- We will assume that the option can be exercised only at a fixed set of dates

$$0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = T.$$

- At any exercise time, we compare the payoff from immediate exercise with the value of continuation. If the immediate payoff is higher, then we exercise
- Thus, our decision is based on the continuation value, i.e. the value of holding rather than exercision the option

$$C(t_i) = \mathbb{E}_{\mathbb{Q}}\left[\frac{U(t_{i+1})}{B(t_{i+1})}\middle|\mathcal{F}_{t_i}\right]B(t_i) = \mathbb{E}_{\mathbb{Q}}\left[U(t_{i+1})\frac{B(t_i)}{B(t_{i+1})}\middle|\mathcal{F}_{t_i}\right].$$

- Estimating these conditional expectations is the main difficulty in pricing American Options by simulation

How to estimate the Continuation Values?

- Note that under the assumption of r constant, we have

$$C(t_i) = \mathbb{E}_{\mathbb{Q}}\left[U(t_{i+1}) \frac{B(t_i)}{B(t_{i+1})} \middle| \mathcal{F}_{t_i} \right] = e^{-r(t_{i+1}-t_i)} \mathbb{E}_{\mathbb{Q}}\left[U(t_{i+1}) \mid \mathcal{F}_{t_i} \right]$$

- The idea is to use regression methods to estimate the continuation values from simulated paths
- Each continuation value $C(t_i)$ is then approximated by the regression of the discounted option value $V(t_i, t_{i+1})$ on the current state $X(t_i)$

Least-Squares Monte Carlo

- Approximate $C(t_i)$ by a linear combination of known functions of the current state $X(t_i)$, i.e.

$$C(t_i) = \sum_{j=0}^{\infty} \alpha_{ij} L_j(X(t_i)), \tag{1}$$

  where $\alpha_{ij} \in \mathbb{R}$ and the functions $L_j(x)$ are the Laguerre polynomials
- Use least-squares method to estimate the coefficients α_{ij} from the pairs

$$(X(t_i, \omega), V(t_i, t_{i+1}, \omega)). \tag{2}$$

- Hence, we get

$$C(t_i) \approx \sum_{j=0}^{\infty} \hat{\alpha}_{ij} L_j(X(t_i)). \tag{3}$$

Remarks

- Note that one can use any other basis functions, e.g. Legendre, Hermite, Chebyshev polynomials. The accuracy would depend on the choice!
- In practice, the approximation is finite (of course)

$$C(t_i) \approx \sum_{j=0}^{M} \hat{\alpha}_{ij} L_j(S(t_i)). \tag{4}$$

- Longstaff and Schwartz (2001) least square Monte Carlo (LSM) method is computationally efficient, converges to the true value, and can be readily extended to high dimensions.

Binomial Trees

- Early attempts made to price American options are the binomial lattice model of Cox, Ross, and Rubinstein (1979).
- Binomial model has been extended by Boyle (1986) in which a middle price jump was incorporated in the price tree. The result trinomial model converges to true option values quicker than that of binomial model.
- Horasanlı (2007) also reached the same conclusion that trinomial model have a faster convergence rate.

Finite Differences

- Finite difference methods were introduced by Schwartz (1977) and Brennan and Schwartz (1977).
- To compensate explicit finite difference's instable problem (not always converges), Hull and White (1990) proposed modified version of explicit finite difference which always converge.

Numerical Approximations

- Geske and Johnson (1984) presented an exact analytic solution to the American put problem. However their formula is an infinite series that can only be evaluated approximately by numerical methods.
- Based on this exact formula, MacMillian (1986) proposed a quadratic approximation of American put option.
- Kim (1990) developed another analytical solution where the American option value represented by an integral equation where the exercise boundary is implicitly defined. So a computationally intensive recursive numerical procedure can be performed to solve for the exercise boundary and option price.

When to Use Monte Carlo?

- Higher dimensions
- Although theoretical multinomial model have been developed, its practical use in higher dimensional problem proved to be very difficult. This difficulty has to do with data storage requirements. Increasing dimensions make the lattice methods and finite differences computationally prohibitive.
- Complex Dynamics

Final Remarks

- Although this topic may seem old, there are still many challenges and it is an active research area
- Speed, speed, speed (we need faster computation on a large scale)
- It is not only about pricing, we also need to calculate derivatives
- I hope I managed to convince you that Quant work is fun!

Thanks for your attention!

A digital version of this presentation and the code that we used today will be available in my GitHub

https://github.com/quantgirluk

If you want to find more resources and tips to become a Quant please take a look at

The Quant Project